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Force theory for multiphase bodies

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Abstract

We present a framework for the study of bodies wherein the deformation gradient may suffer a jump across an evolving nonmaterial interface. To formulate the kinematics relevant to such a situation, we use a global approach in which the configuration space has the structure of an infinite dimensional bundle. We show that a force, defined as an element of the cotangent bundle of the configuration manifold, may be represented by bulk and interfacial stress measures. The invariant decomposition of that force into bulk and interfacial components is discussed and we show that, in the case where the stress measures representing the force are given in terms of smooth densities, such a decomposition is determined by the average stress on the interface.

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1. Introduction

This paper presents a geometric framework for the mechanics of *multiphase bodies* – that is, deformable continua containing moving nonmaterial surfaces across which the deformation, though continuous, suffers a discontinuity in its gradient. In considering such bodies, we are motivated by recent theories for solid–solid phase transitions developed

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by Gurtin and Struthers [1] and Gurtin [2,3], theories that involve a highly nonstandard treatment of forces. [1] Specifically, our goal is to provide an alternate perspective in which certain hypotheses of Gurtin and Struthers arise as consequences of the underlying geometric structure.

Given a multiphase body, we restrict attention to the case in which the differential topological structure of the surface across which its deformation gradient jumps remains constant. In other words, we assume that there exists a two-dimensional manifold, termed the *interface*, and that the various surfaces of discontinuity, termed *configurations of the interface*, are diffeomorphic to it. While any embedding of the interface in the body induces a configuration of the interface, any two embeddings that have the same image will induce the same configuration of the interface. Thus, the collection of configurations of the interface in the body is a *manifold of borders* or a *manifold of shapes* as considered by Kijowski and Komorowski [8], Komorowski [9], Michor [10], and Binz and Fischer [11]. In the terminology of continuum mechanics, the interface lacks material structure.

To overcome the difficulties associated with the lack of material structure we use a general formulation of continuum mechanics suggested in [12,13]. Specifically, we consider an infinite dimensional manifold structure for the configuration space of the body. Generalized velocities and forces are then defined, respectively, as elements of the tangent and cotangent bundles of the configuration space. Being linear functionals, the representation of forces therefore depends on the class of admissible mappings. Natural choices of the admissible mappings and topology allow a representation of forces by measures and an associated generalization of the notion of stress. We note that this approach holds in the case where both the body and the underlying physical space are general differentiable manifolds, and leads, without any reliance on the notion of equilibrium, to proofs of the existence of stress in the case where it may be as irregular as a measure and also an elucidation of certain properties of stress.

Letting Q_κ denote the collection of configurations (of the body in space) that suffer a discontinuity of the derivative at the configuration κ of the interface, the configuration space of the multiphase body is, in our setting, just

$$Q = \bigcup_{\kappa \in Q_I} Q_\kappa,$$

where Q_I is the configuration space of the interface. To apply the approach described above, we take advantage of results due to Kijowski and Komorowski [8] and Komorowski [9] – who work in a context similar to ours – and show how Q may be given the structure of a bundle. For the C^∞ case, Q is smooth. For the C^ω case, with ω finite, TQ possesses the structure of a topological Banach manifold (see [9]).

Ideally, a theory for multiphase bodies should lead to a natural decomposition of generalized forces, and possibly velocities, into components associated with the interface and the bulk. In fact, we are able to – on the basis of a physically well-motivated constitutive assumption – construct a connection on the cotangent bundle that allows a decomposition of

[1] See also the related work of Leo and Sekerka [4], Pitteri [5], Pfenning and Williams [6], and Lusk [7].

forces into interfacial and bulk components. Regarding generalized velocities as elements of the double dual, the decomposition of forces then allows a generalized decomposition of the velocities in which the bulk component is the velocity field of material points. Moreover, we show that, so long as the stress measures representing a force are given in terms of smooth densities, a natural decomposition of the force is available. In this case, the force f on a multiphase body has the representation

$$f(\dot{\chi}) = \int_B (s \cdot v + S \cdot \mathrm{D}v)\,\mathrm{d}V + \int_\kappa (n_\kappa \cdot [\![S \circ \mathrm{D}\chi(n_\kappa)]\!] \dot{u} + c\dot{u} + c \cdot \mathrm{D}\dot{u})\,\mathrm{d}A.$$

Here, v is the velocity field of the material points, $\dot{u}$ is the scalar normal velocity of the interface, s is a vector field representing the ambient bulk-force (which vanishes in equilibrium), S is a tensor field representing the bulk stress, χ is the configuration of the multiphase body, c is a real function on the interface representing the ambient interfacial-force, $\boldsymbol{c}$ is a vector field that represents interfacial shear, n_κ is the interfacial unit normal field, and double-brackets denote jumps of the enclosed quantities across the interface. The representing stress measures are not determined uniquely by the force; if, however, they are given, the force may be restricted to any subbody P of B by restricting the integrals appropriately. Doing so, the restriction f_P of f to P may be represented in the form

$$f_P(\dot{\chi}) = \int_P b \cdot v\,\mathrm{d}V + \int_{\partial P} t \cdot v\,\mathrm{d}A \int_{P \cap \kappa} (\tau \dot{u} + t_\kappa \cdot \langle\!\langle v \rangle\!\rangle)\,\mathrm{d}A + \int_{\partial P \cap \kappa} t_I \dot{u}\,\mathrm{d}L,$$

where $t_I = \boldsymbol{c} \cdot \nu$ is the interfacial boundary force, ν is the unit normal to $\partial P \cap \kappa$ on κ, angled double-brackets denote the average of the enclosed object on the two sides of the interface, $b = s - \operatorname{div} S$ is the bulk body force, $t = S(n_{\partial P})$ is the bulk boundary force, $t_\kappa = [\![S(n_\kappa)]\!]$ is the force exerted on the interface by the adjoining bulk phases, and $\tau = c - \operatorname{div}_\kappa \boldsymbol{c} + t_\kappa \cdot \langle\!\langle \mathrm{D}\chi(\kappa) \rangle\!\rangle$ is the component of the force dual to the normal interfacial velocity field. While the contact forces for a generic subbody P depend on the representing stress fields, the contact and body forces for B depend only on f.

This representation together with the ensuing "balance equations" and "boundary conditions" are analogous to those postulated by Gurtin and Struthers [1], with the difference that our framework, which ignores energetics, precludes any discussion of interfacial tension (or, more generally, interfacial energy).

For the sake of completeness, Section 2 of the paper reviews the mechanics of nonmaterial interfaces (based on [14]). Section 3 discusses the mechanics of *composite bodies* – multiphase bodies for which the interface remains fixed. Nonmaterial surfaces and composite bodies are basic elements in the development of the mechanics of multiphase bodies, whose kinematics and force theory are considered in Sections 4 and 5, respectively. In Section 6 we consider the decomposition of forces into interfacial and bulk components and Section 7 presents the important example in which the stresses are given by smooth densities. Since we rely on the constructions of Kijowski and Komorowski for the configuration spaces, some proofs are omitted for the sake of conciseness.

Our adherence to the use of C^ω topologies, instituted by Kijowski and Komorowski [8] and Komorowski [9] in their constructions of infinite dimensional manifolds of mappings, is suitable to exhibit the geometrical structure and the framework for force and stress theory. By no means do we expect that such topologies would be adequate for the discussion of the partial differential equations governing multiphase bodies. Indeed, to arrive at those partial differential equations would require constitutive equations determining the stresses as functionals of the underlying kinematic variables. Analysis of the ensuing partial differential equations would rest on an investigation of regularity issues and the introduction of appropriate solution spaces (e.g., adequate Sobolev spaces).

2. Review of the mechanics of nonmaterial interfaces

The kinematics and force theory for nonmaterial surfaces, discussed in [14], are basic ingredients in the formulation of the kinematics and force theory for multiphase bodies. Thus, for the sake of completeness we present in this section the relevant definitions, methods, and results of [14] and the references cited therein.

Definition 2.1. A *body*, B, is a compact three-dimensional C^ω, $\omega = 1, \dots, \infty$, manifold, with boundary ∂B, that can be embedded in $\mathbb{R}^3$.

Definition 2.2. An *interface*, I, is a compact orientable C^ω surface in $\mathbb{R}^3$.

Definition 2.3. A *configuration*, κ, *of an interface* I in a body B is a submanifold κ of the interior, $\mathrm{Int}(B)$, of B that is C^ω diffeomorphic to I. We use Q_I to denote the collection of all configurations of I in B.

Definition 2.4. For $X \in \kappa \in Q_I$, the *space of layers* at X is the quotient space $\Lambda_X\kappa = T_X B / T_X\kappa$. The one-dimensional vector bundle

$$\Lambda\kappa = \bigcup_{X\in\kappa} \Lambda_X\kappa$$

with projection $\lambda_\kappa : \Lambda\kappa \to \kappa$, will be referred to as the *bundle of layers*.

Proposition 2.1. (Komorowski [9], Kijowski and Komorowski [8], Michor [10], Binz and Fischer [11]). *The interfacial configuration space Q_I can be endowed with the structure of a manifold modeled on the topological vector space $C^\omega(I)$ of real functions on the interface. For any $\kappa \in Q_I$, we have an isomorphism $T_\kappa Q_I = C^\omega(\lambda_\kappa)$, where $C^\omega(\lambda_\kappa)$ is the space of C^ω sections of λ_κ.*

Remark 2.1. The construction of the manifold structure uses a deformation of $\kappa \in Q_I$ in a tubular neighborhood containing it. A real valued function u defined on κ (or I) can, by displacing the point $X \in \kappa$ to the point in the body whose tubular-neighborhood-coordinates are $(X, u(X))$, represent a neighboring configuration of the interface.

Remark 2.2. Given a metric (induced, for example, by a reference configuration of B in $\mathbb{R}^3$) on B, we will often identify $\dot{u}$ with its component relative to the unit normal vector to κ so that $C^\omega(\lambda_\kappa)$ may be identified with $C^\omega(\kappa)$, the collection of real functions. In this case, the section of λ_κ representing an element $\dot{u} \in T_\kappa Q_I$ is called the *normal velocity field* associated with $\dot{u}$.

Consistent with the general framework described in the introduction, forces on κ are elements of the cotangent space $(T_\kappa Q_I)^* = (C^\omega(\lambda_\kappa))^*$. As such, forces are section distributions (see [15–17]) and, since κ is compact, forces are of finite order.

Definition 2.5. Given k finite, a *force of order k on an evolving interface*, i.e. an *accretive force of order k*, at the configuration κ of I is an element of $(C^k(\lambda_\kappa))^*$.

Remark 2.3. We use the term *accretive* in the sense of Gurtin and Struthers [1], who employ that term to distinguish forces associated with the motion of a nonmaterial interface – whereby material of one phase *grows* at the expense of another – from the more standard Newtonian forces associated with the motion of material particles.

For simplicity, we restrict attention, in the remainder of this section, to forces of order 1.

Proposition 2.2. *Any accretive force $f \in (C^1(\lambda_\kappa))^*$ admits the representation*

$$f(\dot{u}) = \int_\kappa j(\dot{u}) \cdot \mathrm{d}\Xi,$$

where $j: C^1(\lambda_\kappa) \to C^0(J(\lambda_\kappa))$ is the jet extension mapping and Ξ is a measure over κ valued in $J(\lambda_\kappa)^*$, the dual of the jet bundle of the vector bundle λ_κ. For sections $\boldsymbol{e}$ of a vector bundle and $\boldsymbol{f}$ of its dual, $\boldsymbol{e} \cdot \boldsymbol{f}$ denotes the real function $\boldsymbol{f}(X)(\boldsymbol{e}(X))$. Further, if $\boldsymbol{f}$ is a measure valued in the dual bundle, $\boldsymbol{e} \cdot \boldsymbol{f}$ is the analogous real measure. The measure Ξ is referred to as the interfacial stress.

Remark 2.4. In analogy to the usual situation in continuum mechanics, a force on I does not determine a unique stress on I. If a force system – i.e., a force on every submanifold of κ – is given and certain consistency conditions hold, a method due to Segev and de Botton [18] can be used to show that a unique stress may be determined.

If a stress Ξ representing the accretive force $f \in C^1(\lambda_\kappa)^*$ is given, a force f_S of order 1 is induced on any two-dimensional submanifold S of κ by

$$f_S(\dot{u}) = \int_S j(\dot{u}) \cdot \mathrm{d}\Xi.$$

Proposition 2.3. *Given a Riemannian metric on B, any accretive force $f \in C^1(\lambda_\kappa)^*$ may be represented in the form*

$$f(\dot{u}) = \int_{\kappa} \dot{u}\,\mathrm{d}\xi + \int_{\kappa} \mathrm{D}\dot{u}\cdot\mathrm{d}\Xi$$

for all $\dot{u} \in C^1(\lambda_\kappa)$, *where* D *denotes the differentiation operator,* ξ *is a real valued measure over* κ, *and* Ξ *is a measure over* κ *valued in* $T\kappa$. *We will refer to the measure* ξ *as the* ambient (self) accretive force measure *and to* Ξ *as the* accretive stress measure.

Proposition 2.4. *Assume a Riemannian metric is given on* κ. *Then, every accretive force* $f \in C^1(\lambda_\kappa)^*$ *at the configuration* κ *of* I *can be approximated with arbitrary accuracy by the smooth real function* c *over* κ, the ambient accretive force field, *and the smooth vector field* $\mathbf{c}$ *tangent to* κ, *the* accretive shear field, via

$$f(\dot{u}) \cong \int_{\kappa} (c\dot{u} + \mathbf{c}\cdot \mathrm{D}\dot{u})\,\mathrm{d}A,$$

where A *is the area measure on* κ. *If* c *and* $\mathbf{c}$ *are given, then, the force* f_S *for a two-dimensional submanifold with boundary* S *of* κ *is provided by*

$$f_S(\dot{u}) = \int_S b_I\dot{u}\,\mathrm{d}A + \int_{\partial S} t_I\dot{u}\,\mathrm{d}L,$$

where $\operatorname{div}\mathbf{c} + b_I = c$ *on* S *and* $\mathbf{c}\cdot\boldsymbol{\nu} = t_I$ *on* ∂S. *Here,* $\boldsymbol{\nu}$ *is the unit normal on* κ *to* ∂S. *The field* b_I *is the* accretive body force field *and the field* t_I *is the* accretive boundary force.

3. Kinematics and force theory for composite bodies

Before considering the case of an interface that is an evolving nonmaterial surface of discontinuity in the deformation gradient, we discuss the theory of forces for composite bodies – that is, bodies that contain a material interface across which the deformation gradient may jump. [2]

Definition 3.1. A *composition* of a simple body B is a collection of m connected subbodies $\{B_p\}$, $p = 1, \ldots, m$, called *phases*, whose union is B and whose interiors are mutually disjoint. A *composite body* is a body B together with a decomposition of B.

Remark 3.1. Our usage here of the noun "phase" deviates somewhat from what is standard. Using that term in its traditional physical sense, two disconnected regions in a body may be composed of the same "material phase". As we do not consider material properties here, any indication that two disconnected subbodies are composed of the same "material phase" is irrelevant. Hence, to yield a unique decomposition of the body into phases, we require that the phases be connected. Our "phases" may be thought of as "phase connected components".

[2] Here, our approach is based on an example discussed by Segev [19].

Remark 3.2. Clearly, the common boundaries in a decomposition of a body B constitute a configuration of an interface in B. In addition, a configuration of an interface in the body generates a decomposition of B.

Definition 3.2. A *composite configuration* of a composite body B with decomposition $\{B_p\}$ is a continuous mapping $\boldsymbol{y}: B \to \mathbb{R}^3$ that satisfies the following conditions:

(i) For each p, the restriction $\boldsymbol{y}|_{B_p}$ is a C^ω embedding of B_p in $\mathbb{R}^3$.

(ii) If $\partial B_p \cap \partial B_q \neq \emptyset$, $p > q$, the jump of the gradient $[\![\mathrm{D}\boldsymbol{y}]\!]_q^p$, defined on $\partial B_p \cap \partial B_q$ by

$$[\![\mathrm{D}\boldsymbol{y}]\!]_q^p = \mathrm{D}(\boldsymbol{y}|_{B_p}) - \mathrm{D}(\boldsymbol{y}|_{B_q}),$$

is nonzero at each point on the common boundary.

(iii) The mapping $\boldsymbol{y}$ is injective.

Remark 3.3. We recall that the requirement that $\boldsymbol{y}$ is continuous implies that $[\![\mathrm{D}\boldsymbol{y}(\boldsymbol{\ell})]\!]_q^p = 0$ for every vector $\boldsymbol{\ell}$ tangent to the common boundary.

Proposition 3.1. *The configuration space Q_κ of the composite body, i.e., the collection of all composite configurations, is a Banach manifold for a finite ω and otherwise is a Frechet manifold. The tangent space $T_{\boldsymbol{y}} Q_\kappa$ to the configuration space at any configuration $\boldsymbol{y}$ may be identified with*

$$T_\kappa = \{\boldsymbol{v} \in \boldsymbol{C}^0(\boldsymbol{B}, \mathbb{R}^3) : \boldsymbol{v}|_{\boldsymbol{B_p}} \in \boldsymbol{C}^\omega(\boldsymbol{B_p}, \mathbb{R}^3)\},$$

with the topology on T_κ defined using the standard C^ω seminorms on the various phases.

Proof. Let $\tilde{T}_\kappa = \prod_p C^\omega(B_p, \mathbb{R}^3)$ be equipped with the product of the topologies on $C^\omega(B_p, \mathbb{R}^3)$ for $p = 1, \ldots, m$ and consider the mapping $\iota: T_\kappa \to \tilde{T}_\kappa$ given by $\iota(\boldsymbol{v}) = \{\boldsymbol{v}|_{\boldsymbol{B_p}}\}$. Since the configuration space $Q_p = \mathrm{Emb}^\omega(B_p, \mathbb{R}^3)$ of each phase p is open in $C^\omega(B_p, \mathbb{R}^3)$ (see [10,20]), it follows that $\prod_p Q_p$ is open in $\tilde{T}_\kappa$. Further, since the mapping ι is continuous, the inverse image $\iota^{-1}(\prod_p Q_p)$, which is the collection of mappings of the body into space that satisfy condition *(i)* of Definition 3.2, is open in T_κ. We omit the proof that the collection of mappings consistent with *(ii)* and *(iii)* is open in T_κ. □

Remark 3.4. Appealing to Proposition 3.1, we may, for the various $\boldsymbol{y} \in Q_\kappa$, identify $(T_{\boldsymbol{y}} Q_\kappa)^*$ with $(T_\kappa)^*$.

Definition 3.3. A *composite-body* force f of order ω on a composite body at the configuration $\boldsymbol{y}$ is an element of $(T_{\boldsymbol{y}} Q_\kappa)^*$, the dual to the tangent space.

Proposition 3.2. *Every force $f \in T_\kappa^*$ can be represented by a collection $\{f_p\}$, $p = 1, \ldots, m$, with $f_p \in C^\omega(B_p, \mathbb{R}^3)^*$, in the form $f = \iota^*(f_1, \ldots, f_m)$, or alternatively, $f(\dot{\boldsymbol{y}}) = \sum_p f_p(\dot{\boldsymbol{y}}|_{B_p})$, for all $\dot{\boldsymbol{y}} \in T_\kappa$.*

Proof. The proof follows directly from the fact that ι is a linear, continuous injection. □

Remark 3.5. Since $(T_{y|_{B_p}} Q_p)^* = C^{\omega}(B_p, \mathbb{R}^3)^*$, we refer to the various f_p as *phase forces*.

Remark 3.6. Since the mapping

$$\iota: T_\kappa = T_y Q \to \tilde{T}_\kappa = \prod_p C^{\omega}(B_p, \mathbb{R}^3) = \prod_p T_{y|_{B_p}} Q_p$$

of Proposition 3.1 is not surjective (due to the compatibility (i.e., continuity) constraint), its dual mapping,

$$\iota^*: \prod_p (T_{y|_{B_p}} Q_\kappa)^* = \prod_p C^{\omega}(B_p, \mathbb{R}^3)^* \to (T_y Q_p)^*$$

is not injective. Thus, while any force on the composite body may be represented by a collection of simple body forces for the various phases, this representation is not unique. In particular, at the interface, a regular force system will not provide the ambient forces for the distinct phases, but only their sum. (The situation here is similar to that considered in Remark 2.4.)

Remark 3.7. Since each B_p is compact, any element $f_p \in C^{\omega}(B_p, \mathbb{R}^3)^*$, being a distribution with compact support, is of finite order even for the case $\omega = \infty$. Thus, there is a finite k such that $f_p \in C^k(B_p, \mathbb{R}^3)^*$ for all $p = 1, \ldots, m$. For simplicity, we restrict ourselves to the case $k = 1$. The next two propositions consider representations of such forces by measures.

Proposition 3.3. *Every force $f \in (T_\kappa)^*$ can be represented by a collection of measures $\{(\boldsymbol{\sigma}_p, \boldsymbol{\Sigma}_p)\}$, $p = 1, \ldots, m$, in the form*

$$f(\dot{y}) = \sum_{p=1}^{m} \int_{B_p} \dot{y}|_{B_p} \cdot \mathrm{d}\boldsymbol{\sigma}_p + \sum_{p=1}^{m} \int_{B_p} \mathrm{D}(\dot{y}|_{B_p}) \cdot \mathrm{d}\boldsymbol{\Sigma}_p$$

for all $\dot{y} \in T_\kappa$. Here, the measures $\boldsymbol{\sigma}_p$ are valued in $\mathbb{R}^3$ and will be termed the ambient (self) force measures, *and the measures $\boldsymbol{\Sigma}_p$ are valued in $L(\mathbb{R}^3, \mathbb{R}^3)$ and will be termed the* stress tensor measures.

Proof. The proof follows immediately from the representation of the various elements $f_p \in C^1(B_p, \mathbb{R}^3)^*$ by measures (see [18]). □

Remark 3.8. The representation of a phase force by an ambient force and a stress measure is, as discussed in [18], not unique. This nonuniqueness compounds that discussed in Remark 3.6.

Proposition 3.4. *Assume that a reference configuration of the composite body in space is given and identify the body with this reference configuration. Assume, also, that the self-forces $\boldsymbol{\sigma}_p$ and stress tensor measures $\boldsymbol{\Sigma}_p$ that represent a composite force f of order* 1 *are given, respectively, in terms of densities* s_p *and* S_p *smooth with respect to the volume measure on B. Then, for each* $\dot{y} \in T_\kappa$, f *admits a unique representation*

$$f(\dot{y}) = \int_B \boldsymbol{b} \cdot \dot{y} \, \mathrm{d}V + \int_{\partial B} \boldsymbol{t} \cdot \dot{y} \, \mathrm{d}A + \int_\kappa \boldsymbol{t}_\kappa \cdot \dot{y} \, \mathrm{d}A$$

in terms of (piecewise) smooth fields $\boldsymbol{b}$, $\boldsymbol{t}$, *and* $\boldsymbol{t}_\kappa$ *defined by*

$$\begin{aligned} \boldsymbol{b} &= s_p - \operatorname{div} S_p \quad \text{on } B_p, \\ \boldsymbol{t} &= S_p(\boldsymbol{n}_{\partial B}) \quad \text{on } \partial B \cap B_p, \\ \boldsymbol{t}_\kappa &= [\![S(\boldsymbol{n}_\kappa)]\!] \quad \text{on } \kappa, \end{aligned}$$

where $[\![S]\!]$ *is computed using a particular orientation of the interfacial normal.*

Proof. The proof follows directly from Proposition 3.3 and Gauss's theorem. □

Remark 3.9. The forces $\boldsymbol{t}_\kappa$ are those that act on an interface separating the components of a composite body. We may compare these with those derived by Gurtin and Murdoch [21] in their investigation of the mechanics of material surfaces. We note that we are missing a term corresponding to deformational surface stress. Within our framework, such terms will arise if we either consider the interface as a two-dimensional body embedded in B or, alternatively, consider forces of order 2.

In light of Remark 3.6, we now discuss the resolution of the nonunique representation of composite body forces by phase forces.

Definition 3.4. A *system of phase forces* is a right inverse

$$r\colon (T_y Q_p)^* \to \prod_p (T_{y|_{B_p}} Q_P)^* = \prod_p C^\omega(B_p, \mathbb{R}^3)^*$$

of ι^*, providing the set $\{f_p = r(f)_p \mid p = 1, \ldots, m\}$, $f_p \in (T_y Q_p)^*$ for any composite body force $f \in (T_y Q_p)^*$, so that

$$\iota^*\big(r(f)\big)(\dot{y}) = \sum_p r(f)_p(\dot{y}|_{B_p}) = f(\dot{y}).$$

Remark 3.10. The specification of a system of phase forces allows the restriction of a force on a composite body to some of its subbodies, i.e., the phases. This situation is similar to that encountered when the specification of a stress field allows the restriction of a force on a simple body to all its subbodies. Hence, as the specification of a stress field corresponding to a given force is achieved using constitutive assumptions, we regard the specification of a system of phase forces as a constitutive assumption. In Section 7 we show that in the standard case when the stress measures are represented by smooth densities with respect to the volume measure on the composite body – so that a force f is specified uniquely in terms of $\boldsymbol{b}$, $\boldsymbol{t}$, and $\boldsymbol{t}_\kappa$, the additional information required to specify $r(f)$ is the average value of the stress across the interface.

4. Kinematics of multiphase bodies

We now combine the situations described in Sections 2 and 3 and consider a body containing an evolving surface of discontinuity in the deformation gradient. Thus, a configuration of the body is a composite body configuration where the surface of discontinuity is no longer material.

Definition 4.1. Let I be an interface and B a body. A *(coherent) multiphase configuration* of B consists of a configuration $\kappa \in Q_I$ of the interface and a composite body configuration $\chi \in Q_\kappa$.

Since a configuration of I in B induces a decomposition $\{B_p\}$ of B, whenever we mention a decomposition in the sequel, we mean the decomposition induced by the configuration of the interface at hand.

Definition 4.2. The *multiphase configuration space* of a body B with an interface I is

$$Q = \bigcup_{\kappa \in Q_I} Q_\kappa .$$

The natural mapping $\pi: Q \to Q_I$, which we refer to as the *configuration space projection*, assigns $\kappa = \pi(\chi)$ to every $\chi \in Q_\kappa$.

Proposition 4.1. *The multiphase configuration space of a body B and an interface I possesses the structure of a fiber bundle.*

Definition 4.3. Let $N: \kappa_0 \times \mathbb{R} \to W \subset B$ be a tubular neighborhood of κ_0 in B. A *dragging of the domain* B along N is a differentiable mapping $\delta: U \to \mathrm{Diff}^\omega(B)$, with $U \subset C^\omega(\kappa_0)$ a domain of a chart $\psi: U \to Q_I$, $\psi(u) = \{N(X, u(X)) \mid X \in \kappa_0\}$ containing κ_0, that satisfies

(i) $\delta(u)(X) = N(X, u(X))$ for all $X \in \kappa_0$,
(ii) $\delta(0) = 1_B$,
(iii) $\delta(u)(X) \neq X$ only in a neighborhood of κ_0 contained in W.

Overview of the proof of Proposition 4.1. Roughly, by artificially "deforming" the body in the configuration κ_0 of the interface so that κ_0 is taken to κ, a dragging of the domain allows the identification of Q_κ, $\kappa = \delta(\{\kappa_0\})$, with Q_{κ_0} so long as κ is in a neighborhood of κ_0. Such an artificial deformation allows identification of composite body configurations possessing singularities on κ with composite body configurations possessing singularities on κ_0. Kijowski and Komorowski [8] and Komorowski [9] show that for any configuration κ_0 and any tubular neighborhood N there is a dragging δ of the domain. We will outline the construction of the fiber bundle charts on Q. Let κ_0 be a configuration of the interface, N be a tubular neighborhood of κ_0, (U, ψ) be the induced chart in Q_I, and δ be a dragging of the domain. For $\chi_0 \in Q_{\kappa_0}$ and $u \in U$, we consider the mapping $\chi: B \to \mathbb{R}^3$ given by $\chi(X) = \chi_0(\delta(u)^{-1}(X))$. Obviously, χ is a multiphase configuration of B whose gradient is not continuous on $\kappa = \psi(u)$; hence, $\chi \in Q_\kappa$. We define $\Psi: U \times Q_{\kappa_0} \to \pi^{-1}(\psi(U))$ by $\Psi(u, \chi_0) = \chi_0 \circ \delta(u)^{-1}$. Since for any u, $\delta(u)$ is a diffeomorphism of B, the mapping $\Psi(u, \cdot)$ is a diffeomorphism of Q_{κ_0} with $Q_\kappa, \kappa = \psi(u)$. Thus, the mapping $\Psi: U \times Q_{\kappa_0} \to \pi^{-1}(\psi(U))$ generates a local trivialization of the neighborhood $\pi^{-1}(\psi(U))$ in Q. □

Remark 4.1. Additional geometric insight into the local representation of multiphase configurations using a dragging of the domain results from regarding the configuration $\chi: B \to \mathbb{R}^3$ as a section $(1, \chi): B \to B \times \mathbb{R}^3$ of the trivial fiber bundle $\mathrm{pr}_1: B \times \mathbb{R}^3 \to B$. We note that if $y \in Q_{\kappa_0}$ represents the configuration $\chi \in Q_\kappa$, the fact that $y = \chi \circ \delta(u)$, with u being the representative of κ and $\delta(u): B \to B$, implies that in general the mapping $(\delta(u), y) = (1, \chi) \circ \delta(u): B \to B \times \mathbb{R}^3$ is no longer a section. The advantages of using $(\delta(u), y)$ are that it represents both χ and u, its two components are independent, and it has fixed singularities.

$$\begin{array}{ccc} B \times \mathbb{R}^3 & = & B \times \mathbb{R}^3 \\ {\scriptstyle (\delta(u),y)}\uparrow & & \uparrow{\scriptstyle (1,\chi)} \\ B & \xrightarrow{\delta(u)} & B \end{array}$$

Just as elements of $T_\kappa Q_I$ can be represented invariantly by sections of the bundle of layers, it is possible to represent elements of $T_\chi Q$ via sections of a vector bundle.

Definition 4.4. Given $\chi \in Q$, define an equivalence relation ϱ on $C^\omega(B, TB) \times T_\kappa$, $\kappa = \pi(\chi)$, such that $(\boldsymbol{d}_1, \boldsymbol{e}_1)\varrho(\boldsymbol{d}_2, \boldsymbol{e}_2)$, $\boldsymbol{d}_i \in C^\omega(B, TB)$, $\boldsymbol{e}_i \in T_\kappa$, when the difference $(\boldsymbol{d}_2(X) - \boldsymbol{d}_1(X), \boldsymbol{e}_2(X) - \boldsymbol{e}_1(X))$ is tangent to $(1, \chi)(B)$ for any $X \in B$ and tangent to $(1, \chi)(\kappa)$ at any $X \in \kappa$. We set $\tilde{C}^\omega_\chi = (C^\omega(\tau_B) \times T_\kappa)/\varrho$, where $\tau_B: TB \to B$ is the tangent bundle projection and $C^\omega(\tau_B)$ is the space of C^ω tangent vector fields on B.

Proposition 4.2. *The tangent space $T_\chi Q$ is isomorphic to $\tilde{C}^\omega_\chi$.*

Proof. See [9]. □

Definition 4.5. Let $m: \mathbb{R} \to Q$ be a motion of a multiphase body B and for $t \in \mathbb{R}$ let $\{B_p\}$ be the decomposition of B at the configuration $m(t)$. The *material velocity* $v(X)(t)$ of $X \in B_p$ at $t \in \mathbb{R}$ is given by

$$v(X)(t) = \left.\frac{\partial m(s)(X)}{\partial s}\right|_{s=t}.$$

Below we discuss whether the material velocity is well-defined for a given $X \in B$.

Proposition 4.3. *Let $m: \mathbb{R} \to Q$ be a motion such that $m(0) = \chi_0 \in Q_{\kappa_0}$. Let the tangent to this motion at χ_0 be represented by $(\dot{u}, \dot{y}) \in C^\infty(\kappa_0) \times T_{\kappa_0}$ in a chart constructed using a dragging δ of the domain. Then, the material velocity at $X \in B_p$ is given by*

$$v(X)(0) = \dot{y}(X) - \mathrm{D}(\chi_0|_{B_p})(X)\big(\mathrm{D}(\delta)(\dot{u})(X)\big).$$

Proof. The proof follows immediately upon differentiation of the expression for the representation in a chart of Proposition 4.1 with respect to the parameter t. □

The local representation obtained in Proposition 4.3 leads to the following result, the proof of which is omitted for brevity.

Proposition 4.4. *Let Q contain piecewise C^k configurations of the mutliphase body. Then, there is a continuous linear mapping*

$$\varsigma : T_{\chi_0} Q \to \prod_p C^{k-1}(B_p, \mathbb{R}^3)$$

with $\{B_1, \dots, B_m\}$ the decomposition of the body B induced by $\kappa_0 = \pi(\chi_0)$, such that if $X \in B_p$, then $\varsigma(\dot{\chi})(X) = \boldsymbol{v}(X)(0)$. Here $\boldsymbol{v}$ is the material velocity field corresponding to any motion m, such that $m(0) = \chi_0 \in Q_{\kappa_0}$, that represents the tangent vector $\dot{\chi} \in T_{\chi_0} Q$. The mapping ς is injective and its image is closed.

Definition 4.6. Given $\dot{\chi} \in T_{\chi_0} Q$, we will refer to $\boldsymbol{v} = \varsigma(\dot{\chi})$ as the *material velocity field* corresponding to $\dot{\chi}$. We will denote the pth component of $\varsigma(\dot{\chi})$ by $\dot{\chi}_p$. Although at each point on the interface $\partial B_p \cap \partial B_q$ there are, in general, two distinct values of material velocity (depending on whether the limit is taken from the interior of B_p or that of B_q), we will, when no confusion can arise, commit a notational transgression and treat $\boldsymbol{v} = \varsigma(\dot{\chi})$ as a field on B.

Proposition 4.5. *Let $\dot{\chi} \in T_{\chi_0} Q$ be represented under a chart constructed using δ about the configuration $\kappa_0 = \pi(\chi_0)$ by $(0, \chi_0, \dot{u}, \dot{y})$ (as χ_0 is represented by $(0, \chi_0)$ in the chart about κ_0). Then, the jump, $[\![\boldsymbol{v}]\!]^p_q = [\![\varsigma(\dot{\chi})]\!]^p_q$, of the material velocity field across a boundary between phases is given, for each $X \in \kappa_0$, by*

$$[\![\boldsymbol{v}]\!]^p_q(X) = -[\![\mathrm{D}\chi_0(X)]\!]^p_q\big(\mathrm{D}(\delta)(\dot{u})(X)\big) = -[\![\mathrm{D}\chi_0(X)]\!]^p_q\big(N_{,2}(X)\big)\dot{u}(X),$$

where, $N_{,2}$ is the vector field $\kappa_0 \to TB|_{\kappa_0}$ tangent to the "tubular neighborhood lines" at κ_0, i.e.,

$$N_{,2}(X) = \frac{\partial}{\partial s} N(X, s)\big|_{s=0}.$$

Proof. The proof follows directly from Proposition 4.3 and the properties of δ. □

We note that a metric is naturally induced on the body B by its embedding χ_0 in $\mathbb{R}^3$. Hence, we use the normal bundle to generate a tubular neighborhood in a neighborhood of κ_0. It follows that $N_{,2}(X) = \boldsymbol{n}_\kappa(X)$, with $\boldsymbol{n}_\kappa$ the unit normal to κ_0, and that $\dot{u}$ is the component of the normal velocity field along the unit normal. Thus, the following familiar corollary holds.

Corollary 4.1. Let $\boldsymbol{n}_\kappa$ denote the normal to $\kappa_0 = \pi(\chi_0)$. Then, the kinematic relation $[\![\boldsymbol{v}]\!]^p_q = -[\![\mathrm{D}\chi_0(\boldsymbol{n}_\kappa)]\!]^p_q \dot{u}$ must hold.

In addition to the tangent bundle projection $\tau_Q: TQ \to Q$, we consider the tangent $T\pi: TQ \to TQ_I$ to the bundle projection π. This mapping gives the generalized velocity of the interface associated with a generalized velocity of the multiphase body. In the language of geometry, generalized velocities for which the associated velocity of the interface vanishes – i.e., elements $\dot{\chi} \in TQ$ such that $T\pi(\dot{\chi}) = 0 \in T_{\pi(\tau_Q(\dot{\chi}))}Q_I$ – are the *vertical tangent vectors*. We will use VQ to denote the *vertical subbundle* of TQ, i.e.,

$$VQ = \bigcup_{\kappa \in Q_I} T(Q_\kappa) = \bigcup_{\kappa \in Q_I} Q_\kappa \times T_\kappa,$$

while $i: VQ \to TQ$ will denote the inclusion of the vertical subbundle in the tangent of the bundle Q.

When Q is a general bundle, we observe that an invariant decomposition into the third and fourth factors of an element $w \in TQ$ is impossible. While the third factor is defined invariantly by means of $T\pi$, there is generally no invariant method of extracting a vertical vector from $\dot{\chi}$.

5. Forces and stresses for multiphase bodies

In Section 4 a typical tangent space $T_\chi Q$ was identified with the space $\tilde{C}^\omega_\chi$ containing equivalence classes of vector fields defined on B and taking values in $TB \times T\mathbb{R}^3$. Thus, we can make the following definition.

Definition 5.1. A *multiphase force* f of order ω on a multiphase body at the multiphase configuration χ is an element of $(T_\chi Q)^* = (\tilde{C}^\omega_\chi)^*$.

Remark 5.1. As before, explicit mention of the value of ω will be avoided whenever that value does not influence the discussion. Again, if ω is infinite, the fact that B and I are compact implies that f is a distribution of finite order; only that order is not known a priori.

Remark 5.2. For a given trivialization Ψ, by the local representation of elements of $T_\chi Q$ in the form $(\dot{u}, \dot{y}) \in C^\omega(\kappa) \times T_\kappa$, a force $f \in (T_\chi Q)^*$ can be represented in the form $(\theta, g) \in C^\omega(\kappa)^* \times T^*_\kappa$. The virtual power, therefore, admits the local representation

$$f(\dot{\chi}) = \theta(\dot{u}) + g(\dot{y}), \quad \dot{\chi} \in T_\chi Q,$$

where θ is the *accretive local component* of the force under the given trivialization Ψ, and g is the *bulk local component* of the force under Ψ.

To obtain the transformation rule for the accretive and bulk local components of the force we consider two charts ψ_i, Ψ_i, $i = 1, 2$, so that we have representations $(u_i, y_i, \dot{u}_i, \dot{y}_i)$ of a generalized velocity and representations $(u_i, y_i, \theta_i, g_i)$ for the associated force. Hence, setting $h = \psi_2^{-1} \circ \psi_1: \dot{u}_1 \mapsto \dot{u}_2$ and $H = \mathrm{pr}_2 \circ \Psi_2^{-1} \circ \Psi_1: (\dot{u}_1, \dot{y}_1) \mapsto \dot{y}_2$, we have

$$
\begin{aligned}
f(\dot{\chi}) &= \theta_1(\dot{u}_1) + g_1(\dot{y}_1) \\
&= \theta_2(\dot{u}_2) + g_2(\dot{y}_2) \\
&= \theta_2\big(\mathrm{D}h(\dot{u}_1)\big) + g_2\big(H_{,1}(\dot{u}_1) + H_{,2}(\dot{y}_1)\big) \\
&= (\theta_2 \circ \mathrm{D}h + g_2 \circ H_{,1})(\dot{u}_1) + g_2 \circ H_{,2}(\dot{y}_1),
\end{aligned}
$$

where, $H_{,1}$ and $H_{,2}$ are the respective partial derivatives of H. The transformation rules for the accretive and bulk local components of the force are therefore

$$\theta_1 = \theta_2 \circ \mathrm{D}h + g_2 \circ H_{,1}, \quad g_1 = g_2 \circ H_{,2}.$$

While the bulk local component of a force is an invariant quantity, the accretive local component of the force mixes with the bulk local component under a change of chart and, hence, is not an invariant quantity. Intuitively, this results from the power that the bulk local component of the force performs on the extension, via the dragging of the domain, of the interfacial velocity to the bulk.

Remark 5.3. The bundle structure $\pi: Q \to Q_I$, implies that we have two special types of forces: forces on the multiphase body induced by accretive forces and forces on composite bodies induced by forces on multiphase bodies. The map $T_\chi\pi: T_\chi Q \to T_{\pi(\chi)}Q_I$ has a dual mapping $(T_\chi\pi)^*: (T_{\pi(\chi)}Q_I)^* \to (T_\chi Q)^*$ so that if g is an accretive force for the configuration κ of the interface, $(T_\kappa\pi)^*(g)$ is a force on the multiphase body at any χ with $\pi(\chi) = \kappa$. By definition $(T_\kappa\pi)^*(g)(\dot{\chi}) = g\big(T_\kappa\pi(\dot{\chi})\big)$. Thus, $(T_\kappa\pi)^*(g)$ is a force that expends power only for the velocity of the interface. On the other hand, the mapping $i: V_\chi Q \to T_\chi Q$, induces the dual mapping $i^*: (T_\chi Q)^* \to (V_\chi Q)^*$. Since $V_\chi Q$ is isomorphic with T_κ, $\kappa = \pi_Q(\chi)$, the image of a force f on the multiphase body is a force on a composite body whose surface of singularity is κ and $i^*(f)(\dot{y}) = f\big(i(\dot{y})\big)$ is the restriction of the force to velocities of the multiphase body for which the surface of singularity vanishes momentarily. Locally, i^* has the representation $(u, y, \theta, g) \mapsto (u, y, g)$, with $g \in T^*_\kappa$.

Proposition 5.1. *Let f be a multiphase force of order 1 at the multiphase configuration χ. Then, if a bundle chart is given in a neighborhood of χ, f admits the representation*

$$f(\dot{\chi}) = \int_B \dot{y} \cdot \mathrm{d}\sigma + \sum_{p=1}^{m} \int_{B_p} \mathrm{D}(\dot{y}|_{B_p}) \cdot \mathrm{d}\Sigma_p + \int_\kappa \dot{u}\, \mathrm{d}\xi + \int_\kappa \nabla_\kappa \dot{u} \cdot \mathrm{d}\Xi$$

with $\dot{\chi} \in T_\chi Q$, $(\dot{u}, \dot{y})$ the representatives of $\dot{\chi}$ in the given chart, ξ and Ξ measures over κ valued in $\mathbb{R}$ and T_κ, respectively, σ a measure over B and valued in $\mathbb{R}^3$, and Σ_p a measure defined on B_p (for $p = 1, 2, \ldots m$) and valued in $L(\mathbb{R}^3, \mathbb{R}^3)$.

Proof. We simply combine the representation procedures for accretive forces (Proposition 2.3) and composite body forces (Proposition 3.3) together with the representation by accretive and bulk local components of Remark 5.2. Thus, there exist measures ξ and Ξ over κ

valued in $\mathbb{R}$ and T_κ, respectively, such that, θ, the accretive local component of the force is represented by

$$\theta(\dot{u}) = \int_\kappa \dot{u}\,\mathrm{d}\xi + \int_\kappa \nabla_\kappa \dot{u} \cdot \mathrm{d}\boldsymbol{\Xi}.$$

Similarly, there are measures $\boldsymbol{\Sigma}_p$, $p = 1, 2, \ldots, m$ defined on B_p and valued in $L(\mathbb{R}^3, \mathbb{R}^3)$ and a measure $\boldsymbol{\sigma}$ over B valued in $\mathbb{R}^3$ such that, g, the bulk local component of the force is

$$g(\dot{y}) = \int_B \dot{y} \cdot \mathrm{d}\boldsymbol{\sigma} + \sum_{p=1}^{m} \int_{B_p} \mathrm{D}(\dot{y}|_{B_p}) \cdot \mathrm{d}\boldsymbol{\Sigma}_p,$$

which establishes the proposition. □

Remark 5.4. The representation of forces of order 1 by measures makes it possible to restrict the force to measurable subsets whenever the representing measures are given. Thus for the given representing measures we define

$$f_P(\dot{\chi}) = \int_P \dot{y} \cdot \mathrm{d}\boldsymbol{\sigma} + \sum_{p=1}^{m} \int_{B_p \cap P} \mathrm{D}(\dot{y}|_{B_p}) \cdot \mathrm{d}\boldsymbol{\Sigma}_p + \int_{\kappa \cap P} \dot{u}\,\mathrm{d}\xi + \int_{\kappa \cap P} \nabla_\kappa \dot{u} \cdot \mathrm{d}\boldsymbol{\Xi}.$$

Proposition 5.2. *Assume that the body is identified with a given reference configuration and that the measures ξ, $\boldsymbol{\Xi}$, $\boldsymbol{\sigma}$, and $\boldsymbol{\Sigma}_p$ are given respectively in terms of densities c, $\boldsymbol{c}$, $\boldsymbol{s}$, and $\boldsymbol{S}_p$ smooth with respect to the area measure on κ and the volume measure in B. Then, a force f of order 1 admits the representation*

$$f(\dot{\chi}) = \int_B \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\,\mathrm{d}V + \int_{\partial B} \boldsymbol{t} \cdot \dot{\boldsymbol{y}}\,\mathrm{d}A + \int_\kappa (\dot{\boldsymbol{y}} \cdot \boldsymbol{t}_\kappa + b_I \dot{u})\,\mathrm{d}A,$$

where $b_I = c - \operatorname{div} \boldsymbol{c}$ denotes the accretive local component body force, $\boldsymbol{b} = \boldsymbol{s} - \operatorname{div} \boldsymbol{S}_p$ on B_p, $\boldsymbol{t} = \boldsymbol{S}_p(\boldsymbol{n}_{\partial B_p})$ on $\partial B \cap B_p$, and $\boldsymbol{t}_\kappa = [\![\boldsymbol{S}(\boldsymbol{n}_\kappa)]\!]$ on κ (with the jump computed using a particular orientation of $\boldsymbol{n}_\kappa$).

Proof. The proposition follows immediately from the corresponding situations for evolving interfaces and composite bodies. □

Corollary 5.1. If the representing measures are given in terms of smooth densities, the restriction of the force f (of order 1) to the subbody P is represented by

$$f_P(\dot{\chi}) = \int_P \boldsymbol{b} \cdot \dot{\boldsymbol{y}}\,\mathrm{d}V + \int_{\partial P} \boldsymbol{t} \cdot \dot{\boldsymbol{y}}\,\mathrm{d}A + \int_\kappa \boldsymbol{t}_\kappa \cdot \dot{\boldsymbol{y}}\,\mathrm{d}A + \int_{\kappa \cap P} b_I \dot{u}\,\mathrm{d}A + \int_{\partial\kappa \cap P} t_I \dot{u}\,\mathrm{d}L,$$

with $t_I = \boldsymbol{c} \cdot \nu$ the accretive surface force, $\boldsymbol{b} = \boldsymbol{s} - \operatorname{div} \boldsymbol{S}_p$ in $B_p \cap P$, $\boldsymbol{t} = \boldsymbol{S}_p(\boldsymbol{n}_{\partial P})$ on $\partial P \cap B_p$, and $\boldsymbol{t}_\kappa = [\![\boldsymbol{S}(\boldsymbol{n}_\kappa)]\!]$ on $P \cap \kappa$.

6. The decomposition of forces into bulk and accretive components

The discussion of Remark 5.2 makes it clear that, at this level of generality, there is no invariant way to decompose a force on a multiphase body into accretive and bulk components. Granted an invariant decomposition of the velocity $\dot{\chi}$ into $\dot{u}$, a generalized interfacial velocity, and $\dot{y}$, a generalized composite body velocity, a decomposition of force can, however, be obtained. Such a decomposition is a *connection* on the bundle Q.

For a given $\chi \in Q$ with $\pi(\chi) = \kappa$, we have, recalling that i is an injection and $T_\chi \pi$ is a surjection, the diagram

$$V_\chi Q \xrightarrow{i} T_\chi Q \xrightarrow{T_\chi \pi} T_\kappa Q_I,$$

in which $\operatorname{im} i = \ker T_\chi \pi$. A connection is specified by means of an injection $\Gamma\colon T_\kappa Q_I \to T_\chi Q$ satisfying $T_\chi \pi \circ \Gamma = 1$. The mapping Γ will be referred to as the *connection mapping*. Using charts, the connection mapping can be represented in the form $\dot{u} \mapsto (\dot{u}, \gamma_\chi(\dot{u}))$, where γ_χ is a mapping defined on $T_\kappa Q_I$. The connection mapping induces a mapping $\Delta\colon T_\chi Q \to V_\chi Q$ given by $\Delta = 1 - \Gamma \circ T_\chi \pi$. We refer to Δ as the *vertical projection* induced by Γ. In terms of local representatives $\Delta(\dot{\chi})$ is given by $(0, \dot{y} - \gamma_\chi(\dot{u}))$, where $(\dot{u}, \dot{y})$ are the representatives of $\dot{\chi}$, so that $\Delta(\dot{\chi})$ is indeed in the vertical bundle. We note that $\Delta \circ i = 1$, $\operatorname{im} \Gamma = \ker \Delta$ and that Δ is surjective. The situation is illustrated in the diagram:

$$\begin{array}{ccccc}
V_\chi Q & \xrightarrow{\;\;i\;\;} & T_\chi Q & \xrightarrow{\;T_\chi \pi\;} & T_\kappa Q_I \\
\| & & \| & & \| \\
V_\chi Q & \xleftarrow{\;\;\Delta\;\;} & T_\chi Q & \xleftarrow{\;\;\Gamma\;\;} & T_\kappa Q_I
\end{array}$$

It follows that $(T_\chi \pi, \Delta)\colon T_\chi Q \to T_\kappa Q_I \times V_\chi Q$, whose inverse is $\Gamma + i$, is the required decomposition of the tangent space. The decomposition of the tangent space in terms of the connection mapping induces a decomposition of the cotangent space – the space of forces – using the duals of the corresponding mappings. Thus, if a connection mapping is available we have the diagram:

$$\begin{array}{ccccc}
(V_\chi Q)^* & \xleftarrow{\;\;i^*\;\;} & (T_\chi Q)^* & \xleftarrow{\;(T_\chi \pi)^*\;} & (T_\kappa Q_I)^* \\
\| & & \| & & \| \\
(V_\chi Q)^* & \xrightarrow{\;\;\Delta^*\;\;} & (T_\chi Q)^* & \xrightarrow{\;(\Gamma)^*\;} & (T_\kappa Q_I)^*
\end{array}$$

Here, $(T_\chi \pi)^*$ and Δ^* are injections, Γ^* and i^* are surjections, $\operatorname{im}(T_\chi \pi)^* = \ker i^*$, $i^* \circ \Delta^* = 1$, $\Gamma^* \circ (T_\chi \pi)^* = 1$, $(1 - \Delta^* \circ i^*)(f) \in \operatorname{im}(T_\chi \pi)^*$ for all $f \in (T_\chi Q)^*$ and $(T_\chi \pi)^* \circ \Gamma^* = (1 - \Delta^* \circ i^*)$. It follows that $(\Gamma^*, i^*) : (T_\chi Q)^* \to (T_\kappa Q_I)^* \times (V_\chi Q)^*$ is an isomorphism with inverse $(T_\chi \pi)^* + \Delta^*$ so that

$$f(\dot{\chi}) = i^*(f)(\Delta(\dot{\chi})) + \Gamma^*(f)(T_\chi \pi(\dot{\chi})).$$

Note that the situation is completely analogous to that involving TQ, except that the various mappings have reversed their directions. Therefore, to decompose forces, it suffices to give an injection $\Delta^* : (V_\chi Q)^* \to (T_\chi Q)^*$ such that $i^* \circ \Delta^* = 1$ and set $\Gamma^* =$

$(T_\chi \pi)^{*-1} \circ (1 - \Delta^* \circ i^*)$. The definition of Γ^* is sensible because $(1 - \Delta^* \circ i^*)(f)$ is an element of $\mathrm{im}(T_\chi \pi)^*$. In the case where the topological vector spaces under consideration are reflexive (e.g., finite dimensional spaces) it is possible to obtain Γ by taking the dual of Γ^*, and thus to obtain a decomposition of the virtual velocities. If the relevant spaces are not reflexive, it is possible, by taking the dual again, to decompose the generalized velocities only in a generalized sense – the vertical component will be a member of $(T_\chi Q)^{**}$ rather than of $T_\chi Q$ (see the following commutative diagram).

$$\begin{array}{ccccc}
(V_\chi Q)^{**} & \xleftarrow{i^{**}} & (T_\chi Q)^{**} & \xleftarrow{(T_\chi \pi)^{**}} & (T_\kappa Q_I)^{**} \\
\| & & \| & & \| \\
(V_\chi Q)^{**} & \xrightarrow{\Delta^{**}} & (T_\chi Q)^{**} & \xrightarrow{(\Gamma)^{**}} & (T_\kappa Q_I)^{**}
\end{array}$$

Definition 6.1. A *local dual connection mapping* at $\chi \in Q$ is a mapping $\Delta^*: (V_\chi Q)^* \to (T_\chi Q)^*$ such that $i^* \circ \Delta^* = 1$. Given a force $f \in (T_\chi Q)^*$, we will refer to $i^*(f)$ as the *vertical component of the force* and to $\Gamma^*(f)$ as the *horizontal component of the force.*

Remark 6.1. The foregoing discussion clearly holds for any mechanical system for which the configuration space has the structure of a fiber bundle $\pi: Q \to Q_I$ and thus pertains to more than the mechanics of multiphase bodies.

Corollary 6.1. In the decomposition of a force induced by a local dual connection the bulk (vertical) component f_B of the force f is $f_B(\dot{y}) = i^*(f)(\dot{y}) = f(i(\dot{y}))$ and the accretive (horizontal) component f_I in $(T_\kappa Q_I)^*$ of the force is given by

$$f_I(\dot{u}) = \Gamma^*(f)(\dot{u}) = \left((T_\chi \pi)^{*-1} \circ (1 - \Delta^* \circ i^*)(f)\right)(\dot{u}).$$

Note that, in contrast with the local representatives (θ, g), we do not describe f_I and f_B as "local" components of the force.

Proposition 6.1. *For the C^∞ case, a system of phase forces (see Definition* 3.4)

$$r: (T_\kappa^\infty)^* \to \prod_p C^\infty(B_p, \mathbb{R}^3)^*$$

with $r(g)(\iota(\dot{y})) = g(\dot{y})$, generates a decomposition of forces in $(T_\chi Q)^*$, with $\kappa = \pi(\chi)$, into accretive and bulk components.

Proof. Given r we define a local dual connection mapping by setting

$$\Delta^*(g)(\dot{\chi}) = r(g)(\varsigma(\dot{\chi})), \quad \dot{\chi} \in T_\chi Q,$$

where $\varsigma(\dot{\chi})$ is the restriction of $\dot{\chi}$ into the smooth material velocity fields on the various phases defined in Proposition 4.4. We have

$$(i^* \circ \Delta^*)(g)(\dot{y}) = \Delta^*(g)(i(\dot{y})) = g(\dot{y}), \quad \dot{y} \in V_\chi Q,$$

so that Δ^* is indeed a right inverse of i^*. □

Remark 6.2. Since any force of a finite order is also a force of order ∞, the above decomposition holds for the action of a (multiphase body) force of any order on piecewise C^∞ velocities.

Proposition 6.2. *If a force f is given in a chart by the components (θ, g), then the bulk component of the force, f_B, is represented by $(0, g)$ and the accretive component of the force, f_I, is represented by*

$$\theta(\dot{u}) + \sum_p r(f)_p(\mathrm{D}\chi|_{B_p})(\mathrm{D}\delta(\dot{u})).$$

Proposition 6.3. *Let a dual connection mapping be given by a system of phase forces r as in Proposition 6.1 and denote by v^{**} the image of an element v of the Banachable space $\mathcal{V}$ under the natural inclusion $\mathcal{V} \to \mathcal{V}^{**}$. Then,*

$$\begin{aligned} \Gamma^{**}(\dot{u}^{**})(f) &= f_I(\dot{u}) \quad \text{for all } \dot{u} \in T_\kappa Q_I, \\ \iota^{**} \circ \Delta^{**}(\dot{\chi}^{**}) &= \varsigma(\dot{\chi})^{**} \quad \text{for all } \dot{\chi} \in T_\chi Q, \end{aligned}$$

where the second equality holds in a generalized sense, meaning that

$$\iota^{**} \circ \Delta^{**}(\dot{\chi}^{**})(\hat{g}) = \hat{g}(\varsigma(\dot{\chi})) = \sum_{p=1}^{m} \hat{g}_p(\varsigma(\dot{\chi})_p)$$

for all $\hat{g} \in \prod_{p=1}^{m} (T_\chi|_{B_p} Q_{B_p})^*$.

Hence, the generalized interfacial component of the velocity is indeed the interfacial normal velocity and the generalized bulk component of the velocity is the material velocity represented as an element of $(T_\chi Q)^{**}$. One therefore obtains the expressions

$$\begin{aligned} f(\dot{\chi}) &= \Delta^{**}(\dot{\chi}^{**})(f_B) + T_\chi\pi^{**}(\dot{\chi}^{**})(f_I) \\ &= r(i^*(f))(\varsigma(\dot{\chi})) + f_I(T_\chi\pi(\dot{\chi})). \end{aligned}$$

Propositions 6.2 and 6.3 follow from the definitions of r and the bulk and accretive components of the force.

7. The case of stresses given by smooth fields

We now present an example of the foregoing procedure for the decomposition of forces. The procedure is applied to the class of composite body forces of order 1 whose representing measures are given in terms of smooth stress densities. (In other words, the mapping r is defined only on a subspace of T^*Q.) We emphasize that most continuum-level work on multiphase bodies is concerned with solely this situation (cf. [2–6]).

Thus, identifying B with a reference configuration and fixing our attention on a fixed configuration κ of the interface in the body, we assume that the composite body force g may be represented by piecewise smooth (not necessarily unique) fields $s \in \prod_p C^\infty(B_p, \mathbb{R}^3)$ and $S \in \prod_p C^\infty(B_p, L(\mathbb{R}^3, \mathbb{R}^3))$ through

$$g(\dot{y}) = \sum_{p=1}^{m} \int_{B_p} s_p \cdot (\dot{y}|_{B_p})\, \mathrm{d}V + \sum_{p=1}^{m} \int_{B_p} S_p \cdot \mathrm{D}(\dot{y}|_{B_p})\, \mathrm{d}V.$$

Clearly, if the stress measures σ and Σ representing the bulk local component g of the force f on the multiphase body in a chart on Q may be given in terms of smooth densities s and S (cf. Proposition 5.2), then $i^*(f)$ may, as assumed, be represented by smooth densities. While the foregoing representation depends on what chart we use on Q, we note that whenever a stress representing the bulk local component of a force in one particular chart is given in terms of a smooth density, any other stress representing the bulk local component of the force with respect to any other chart on Q must also be given in terms of a smooth density.

We can now define r by

$$r(g)_p(\dot{y}_p) = \int_{B_p} (s_p \cdot \dot{y}_p + S_p \cdot \mathrm{D}\dot{y}_p)\, \mathrm{d}V, \quad \dot{y}_p \in C^\infty(B_p, \mathbb{R}^3).$$

It follows that the action of the interfacial component of the force is given by

$$f_I(\dot{u}) = \theta(\dot{u}) + \sum_{p=1}^{m} \int_{B_p} \left\{ s_p \cdot \left(\mathrm{D}\chi_p \circ \mathrm{D}\delta(\dot{u})\right) + S_p \cdot \mathrm{D}\left(\mathrm{D}\chi_p \circ \mathrm{D}\delta(\dot{u})\right) \right\} \mathrm{d}V.$$

We observe that the sum on the right is the local expression for the connection mapping. It depends on the dragging of the domain in such a way that when combined with θ the resulting power $f_I(\dot{u})$ is invariant. Recalling (cf. Definition 4.3) that the dragging of the domain is supported in a tubular neighborhood of the interface and that it is possible to construct a dragging of the domain whose support is, in any neighborhood of the interface, contained in a tubular neighborhood, we may choose a sequence of draggings of the domain δ_i, $i = 1, \ldots,$ such that the support of δ_i is contained in the neighborhood $E_i = N(\{\kappa, [-1/i, 1/i]\})$. The left hand side of the last equation does not change with i while the last two sums approach a limit as will be shown below. Hence, if θ_i is the local accretive component of the force f in the chart induced by the dragging δ_i of the domain, then $\theta_i(\dot{u})$ approaches a limit that we will denote by $\hat{\theta}$. Clearly, $\hat{\theta}$ is invariant.

We now compute the limit of the two sums using a normal tubular neighborhood. By the properties of the dragging of the domain, the support of $\mathrm{D}\delta_i(\dot{u})$ is also contained in the neighborhood E_i and $\mathrm{D}\delta_i(\dot{u})(X) = \dot{u}(X)\boldsymbol{n}_\kappa(X)$ for every $X \in \kappa$. Hence, since $\mathrm{D}\chi_p(\mathrm{D}\delta(\dot{u}))$ is bounded and the measure of its support tends to zero:

$$\lim_{i\to\infty} \int_{B_p} s_p \cdot \left(\mathrm{D}\chi_p \circ \mathrm{D}\delta_i(\dot{u})\right) \mathrm{d}V = 0.$$

From the assumption that the densities S_p are differentiable it follows that

$$\lim_{i\to\infty}\sum_{p=1}^{m}\int_{B_p} S_p\cdot\mathrm{D}\big(\mathrm{D}\chi_p\circ\mathrm{D}\delta_i(\dot u)\big)\,\mathrm{d}V=\int_{\kappa} \boldsymbol{n}_\kappa\cdot[\![S\circ\mathrm{D}\chi(\boldsymbol{n}_\kappa)]\!]\dot u\,\mathrm{d}A,$$

where, here and in the sequel, a particular orientation for $\boldsymbol{n}_\kappa$ is chosen (so that $\boldsymbol{n}_{\partial B_q}$ and $\boldsymbol{n}_\kappa$ coincide) and the jump is defined accordingly.

We conclude therefore that

$$f_I(\dot u)=\hat\theta(\dot u)+\int_{\kappa} \boldsymbol{n}_\kappa\cdot[\![S\circ\mathrm{D}\chi(\boldsymbol{n}_\kappa)]\!]\dot u\ \mathrm{d}A,$$

which, in light of the relation $[\![v_1v_2]\!]=\langle\!\langle v_1\rangle\!\rangle[\![v_2]\!]+[\![v_1]\!]\langle\!\langle v_2\rangle\!\rangle$, that holds for any two mappings v_1 and v_2 in $\prod_p C^0(B_p,\mathbb{R}^3)$, can be rewritten as

$$f_I(\dot u)=\hat\theta(\dot u)+\int_{\kappa}\big([\![\boldsymbol{S}(\boldsymbol{n}_\kappa)]\!]\cdot\langle\!\langle\mathrm{D}\chi\rangle\!\rangle(\boldsymbol{n}_\kappa)+\langle\!\langle\boldsymbol{S}(\boldsymbol{n}_\kappa)\rangle\!\rangle\cdot[\![\mathrm{D}\chi(\boldsymbol{n}_\kappa)]\!]\big)\dot u\,\mathrm{d}A.$$

The representation of the power expended in bulk is

$$\Delta^{**}(\dot\chi)(f_B)=r(i^*(f))(\varsigma(\dot\chi))=\sum_{p=1}^{m}\int_{B_p}\big(\boldsymbol{s}_p\cdot\boldsymbol{v}_p+\boldsymbol{S}_p\cdot\mathrm{D}(\boldsymbol{v}_p)\big)\,\mathrm{d}V,$$

where $\boldsymbol{v}=\varsigma(\dot\chi)$ denotes the material velocity field. On appealing to the smoothness of $\boldsymbol{s}_p$ and $\boldsymbol{S}_p$ and Corollary 4.1, this expression leads to

$$\Delta^{**}(\dot\chi)(f_B)=\int_B \boldsymbol{b}\cdot\boldsymbol{v}\,\mathrm{d}V+\int_{\partial B}\boldsymbol{t}\cdot\boldsymbol{v}\,\mathrm{d}A$$
$$+\int_{\kappa}\big([\![\boldsymbol{S}(\boldsymbol{n}_\kappa)]\!]\cdot\langle\!\langle\boldsymbol{v}\rangle\!\rangle-\langle\!\langle\boldsymbol{S}(\boldsymbol{n}_\kappa)\rangle\!\rangle\cdot[\![\mathrm{D}\chi(\boldsymbol{n}_\kappa)]\!]\dot u\big)\,\mathrm{d}A.$$

Here, $\boldsymbol{b}=\sum_p(\boldsymbol{s}_p^*-\operatorname{div}\boldsymbol{S}_p^*)$, with $\boldsymbol{s}_p^*$ and $\boldsymbol{S}_p^*$ the zero extensions of $\boldsymbol{s}_p$ and $\boldsymbol{S}_p$ to B, and $\boldsymbol{t}=\sum_p\boldsymbol{t}_p^*$, where $\boldsymbol{t}_p^*$ is the zero extension to ∂B of $\boldsymbol{S}_p(\boldsymbol{n}_{\partial B_p})$. Combining this expression with the expression for the interfacial power expenditure and assuming that the interfacial forces can be represented by measures given in terms of the smooth densities c and $\boldsymbol{c}$ (allowing us to use the expressions obtained in Proposition 2.4), we obtain the following proposition.

Proposition 7.1. *If a force on a multiphase body can be represented by smooth stress measures, then that force admits the unique representation*

$$f(\dot\chi)=\int_B\boldsymbol{b}\cdot\boldsymbol{v}\,\mathrm{d}V+\int_{\partial B}\boldsymbol{t}\cdot\boldsymbol{v}\,\mathrm{d}A+\int_{\kappa}(\tau\dot u+\boldsymbol{t}_\kappa\cdot\langle\!\langle\boldsymbol{v}\rangle\!\rangle)\,\mathrm{d}A$$

in terms of the body force $\boldsymbol{b}$, the surface force $\boldsymbol{t}$, and the fields τ and $\boldsymbol{t}_\kappa$ defined on κ and valued, respectively, in $\mathbb{R}$ and $\mathbb{R}^3$. Further, for f to be represented by piecewise smooth stress measures $\boldsymbol{s}$, $\boldsymbol{S}$, c, and $\boldsymbol{c}$, we must have

$$\begin{aligned} b &= s - \operatorname{div} S \quad \text{on } B, \\ t &= S(n_{\partial B}) \quad \text{on } \partial B, \\ t_\kappa &= [\![S(n_\kappa)]\!] \quad \text{on } \kappa, \\ \tau &= c - \operatorname{div}_\kappa c + t_\kappa \cdot \langle\!\langle D\chi(n_\kappa) \rangle\!\rangle \quad \text{on } \kappa. \end{aligned}$$

Given a subbody $P \subset B$ whose intersection with κ is a submanifold of κ, the force f_P obtained by the restriction of the various measures to P admits the representation

$$f_P(\dot{\chi}) = \int_P b \cdot v \, \mathrm{d}V + \int_{\partial P} t \cdot v \, \mathrm{d}A \int_{P \cap \kappa} (\tau \dot{u} + t_\kappa \cdot \langle\!\langle v \rangle\!\rangle) \, \mathrm{d}A + \int_{\partial P \cap \kappa} t_l \dot{u} \, \mathrm{d}L,$$

where we now have the additional boundary condition $t_l = c \cdot \nu$ on $\partial P \cap \kappa$ and the remaining fields satisfy the equations and boundary conditions determined above.

Remark 7.1. From the expressions for the representations of the interfacial and bulk components of the force, we conclude that if a force is given by the fields τ, b, t, and t_κ, the additional information needed to decompose that force into these components is the average stress on the interface.

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